IOOpt: Automatic Derivation of I/O Complexity Bounds for Affine Programs

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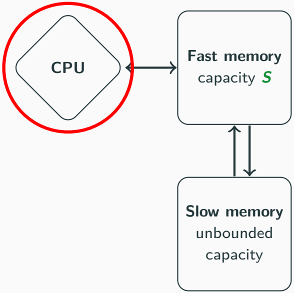
What is I/O complexity?

- Arithmetic complexity = # of operations



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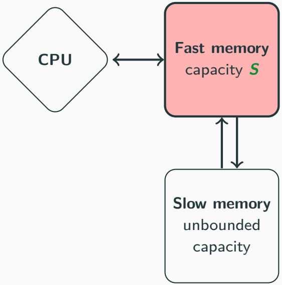

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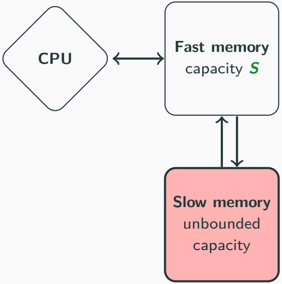
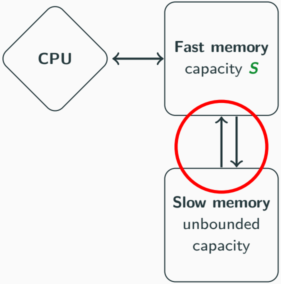


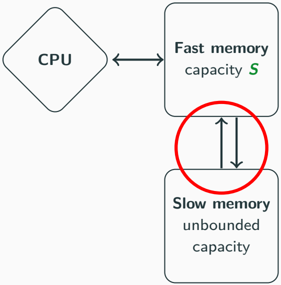
What is I/O complexity?

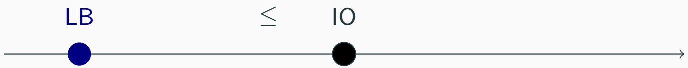
- Arithmetic complexity = # of operations
- I/O cost (schedule-dependent) = amount of data moved between fast and slow memory



What is I/O complexity?

- Arithmetic complexity = # of operations
- I/O cost (schedule-dependent) = amount of data moved between fast and slow memory
- I/O complexity = minimum cost over all schedules





IOLB *(PLDI '20)*
Automated lower
bound computation

Automated Derivation of Parametric Data Movement Lower Bounds for Affine Programs*

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Abstract

Researchers and practitioners have for long worked on improving the computational complexity of algorithms, focusing on reducing the number of operations needed to perform a computation. However the hardware trend nowadays clearly shows a higher performance and energy cost for data movements than computations: quality algorithms have to minimize data movements as much as possible.

The theoretical operational complexity of an algorithm is a function of the total number of operations that must be executed, regardless of the order in which they will actually be executed. But theoretical data movement (or, I/O) complexity is fundamentally different: one must consider all possible legal schedules of the operations to determine the minimal number of data movements achievable, a major theoretical challenge. I/O complexity has been studied via complex manual proofs, *e.g.*, refined from $\Omega(n^3/\sqrt{S})$ for matrix-multiply on a cache size S by Hong & Kung to $2n^3/\sqrt{S}$ by Smith et al. While asymptotic complexity may be sufficient to compare I/O potential between broadly different algorithms, the accuracy of the reasoning depends on the tightness of these I/O lower bounds. Precisely, exposing constants is essential to enable precise comparison between different algorithms: for example the $2n^3/\sqrt{S}$ lower bound allows to demonstrate the optimality of panel-panel tiling for matrix-multiplication.

We present the first static analysis to automatically derive non-asymptotic parametric expressions of data movement lower bounds with scaling constants, for arbitrary affine computations. Our approach is fully automatic, assisting algorithm designers to reason about I/O complexity and make educated decisions about algorithmic alternatives.

CCS Concepts: **• Theory of computation → Design and analysis of algorithms; • Software and its engineering → Automated static analysis.**

Keywords: Data access complexity; I/O lower bounds; Static analysis; Affine programs

1 Introduction

The performance impact of operations and data movement latencies in current architectures can often be effectively masked by using hardware-pipelined implementations. But the volume of data movements required by even an idealized implementation of an algorithm will impose fundamental limits: any implementation of that algorithm will have its performance and energy requirements bounded by this limit [12, 18, 25, 26, 28, 29]. Providing algorithm designers with tools to characterize this fundamental limit is crucial.

Memory movements can be efficiently tracked for a particular algorithm *implementation*, and it is standard practice for performance debugging [1]: Hardware counters can be used to measure cache misses and data traffic. But two different implementations of the same algorithm may have dramatically different memory movement profiles: for example a carefully tiled implementation of matrix multiplication would significantly reduce cache misses versus a naive, untiled one.

In general, determining whether an implementation is sub-optimal or whether the fundamental nature of the algorithm is the limiting factor for the observed cache miss count is crucial. We propose an automatic system to answer this question, potentially alleviating the need for the algorithm designers to produce a concrete optimized implementation. As we specifically target the production of non-asymptotic I/O lower bounds, our system also makes it possible for performance experts to reason about the optimality of their implementations with respect to data movement.

Our fully implemented framework IOLB (for *I/O Lower Bounds*) automatically derives parametric lower bounds with

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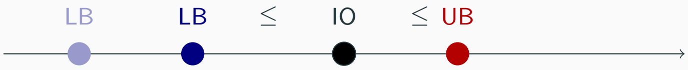
Lower and Upper Bounds



IOLB *(PLDI '20)*
Automated lower bound computation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Data Movement Lower Bounds for Affine Programs

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IOOpt (This paper)

- Improvement of the lower bound algorithm
- Automated upper bound derivation (IOUB)

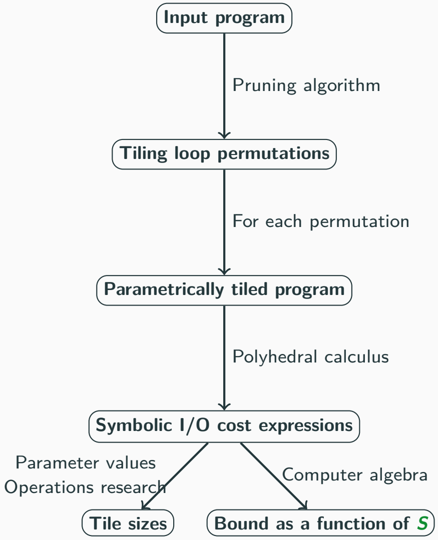
I/O Upper Bounds

I/O complexity upper bound $\Leftrightarrow$ Cost of a particular valid schedule

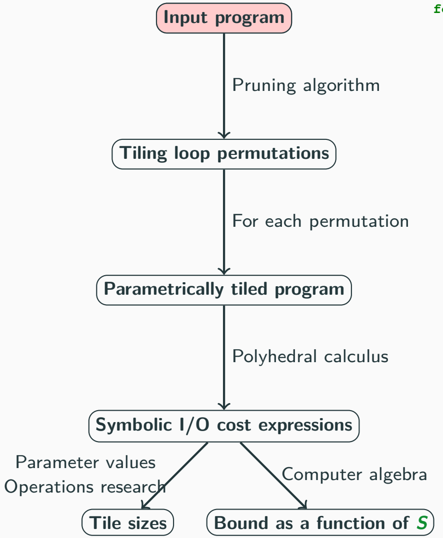
Untiled matrix multiplication	Tiled matrix multiplication
I/O cost: $O(N^3)$	I/O cost: $O(\frac{N^3}{\sqrt{S}})$

$\rightarrow$ How to automatically compute I/O cost for a given schedule?

Upper bound derivation

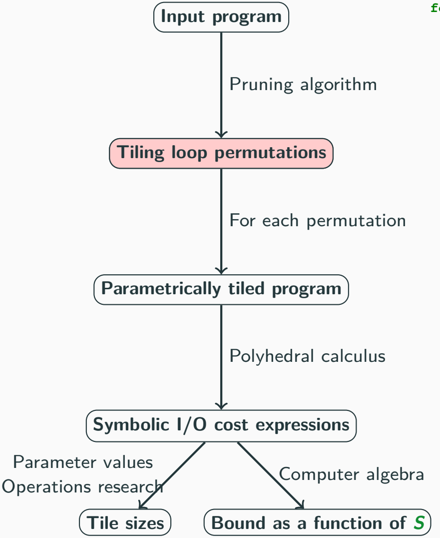


Upper bound derivation




```
for(i = 0; i < Ni; i++)
  for(j = 0; j < Nj; j++)
    for(k = 0; k < Nk; k++)
      C[i][j] += A[i][k] * B[k][j];
```

Upper bound derivation

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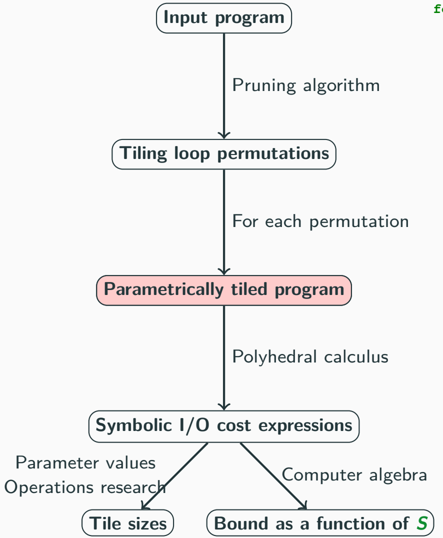
$$\{(i,j,k),(i,k,j),(k,j,i)\}$$

Upper bound derivation

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$\{\boxed{(i,j,k)}, (i,k,j), (k,j,i)\}$

```
for(i1 = 0; i1 < Ni; i1+=Ti)
  for(j1 = 0; j1 < Nj; j1+=Tj)
    for(k = 0; k < Nk; k++)
      for(i = i1; i < i1+Ti; i++)
        for(j = j1; j < j1+Tj; j++)
          C[i][j] += A[i][k] * B[k][j];
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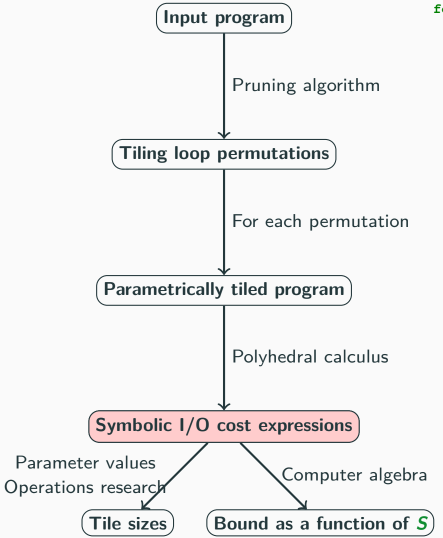
Upper bound derivation

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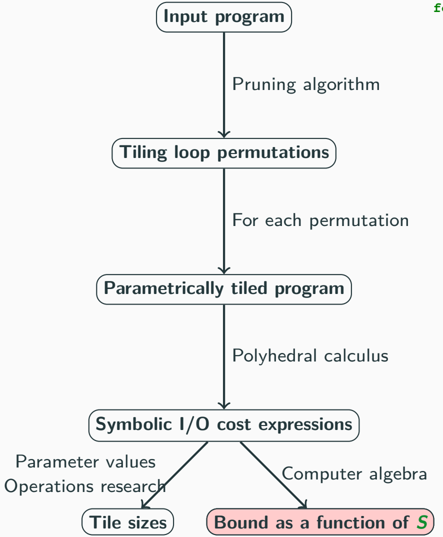
$$IO = N_i N_j N_k \left(\frac{1}{T_i} + \frac{1}{T_j} + \frac{1}{N_k}\right)$$
$$T_i T_j + T_i + T_j \leq S$$

Upper bound derivation

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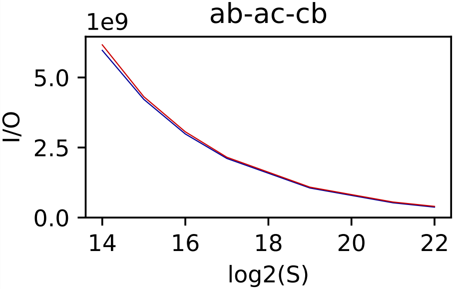
$$UB = N_i N_j \left(\frac{2N_k}{\sqrt{S+1}-1} + 1\right)$$

$$\frac{2N_iN_j(N_k-1)}{\sqrt{S}} \leq IO_{mm} \leq \frac{2N_iN_jN_k}{\sqrt{S+1}-1}$$



In the paper: Analytical results on several convolutions (Yolo9000) and tensor contractions (TCCG), with matching lower and upper bounds

Thank you!